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SOME TESTS FOR A SHIFT IN THE MEAN OF A NORMAL DISTRIBUTION OCCURRING AT UNKNOWN TIME POINTS

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1. Background

Detecting a change in the mean of a sequence of independent normally distributed observations, when the time the change occurs is unknown, is an important statistical problem and has been considered by Vage (1955) and Bhatcharayya and Johnson (1968) using non-parametric procedures, while Chernoff and Zacks (1966), Kander and Zacks (1966), and Mustafi (1968) developed a Bayesian approach. Cox (1961), considered the problem as an example for separate families of hypotheses.

By assuming the time when the change occurs as known and constructing the corresponding likelihood-ratio test for a change in the mean, then averaging the test statistics over all possible shift points, the resulting statistic is used to detect a shift in the mean, when the time the change occurs is unknown. The critical region and power function of the test is derived for one and two-sided alternatives. If a change occurs, it is assumed that it occurs only once.

Consider a test of

$$H_0: X_1, X_2, \ldots, X_n \sim \text{nid}\ (\theta_0, \sigma^2)$$

versus

$$H_1: X_1, X_2, \ldots, X_m \sim \text{nid}\ (\theta_0, \sigma^2)$$

$$X_{m+1}, \ldots, X_n \sim \text{nid}\ (\theta_1, \sigma^2)$$

where σ^2 is the known common variance of the observations, θ_1 and $m (1 \leq m \leq n-1)$ are unknown, but θ_0 may be either known or not. θ_0 is inferred to as initial mean and m the shift point. If $\theta_1 > \theta_1$, H_1 is a one-sided alternative and if $\theta_1 \neq \theta_0$ a two-sided one. For convenience we let $\sigma^2 = 1$.

2. **One-sided alternative, $\theta_1 > \theta_0$.**

Considering θ_0 as known, and the shift point at m=s, the LRT of

$$\begin{array}{ll} H_{0s}: X_{s+1}, \ldots, X_n \sim n(\theta_0, 1) & \\ \text{versus} & (s = 1,2,\ldots,n\text{-}1) \\ H_{1s}: X_{s+1}, \ldots, X_n \sim n(\theta_0, 1) & \end{array}$$

leads to the test statistic

$$\lambda^{(s)} = (n-s)^{-1/2} (\bar{X}_{s+1,n} - \theta_0) \quad (s = 1, 2 \ldots, n-1)$$

where $\bar{X}_{s+1,n}$ is the mean of the last (n-s) observations, and H_{0s} is rejected for large values. Averaging the $\lambda^{(s)}$ over the possible shift points, we have (neglecting a divisor of (n-1))

$$\begin{aligned} T^{(1)} &= \sum_{s=1}^{n-1} \lambda^{(s)} \\ &= \sum_{i=2}^{n} \Big[\sum_{j=1}^{j=i-1} (n-j)^{-1/2} \Big] (x_i - \theta_0), \end{aligned} \tag{2.1}$$

and H_0 is rejected with a type I error of α whenever $T^{(1)} \geq K^{(1)}(\alpha)$, where

$$\alpha = P[T^{(1)} \geq K^{(1)}(\alpha) | H_0].$$

It is easy to verify that the distribution of $T^{(1)}$ under H_0 is normal with mean 0 and variance

$$\mathrm{Var}[T^{(1)} | H_0] = \sum_{i=2}^{n} \Big[\sum_{j=1}^{j=i-1} (n-j)^{-1/2} \Big]^2$$

and under H_1 with mean

$$\mu^{(1)}(m, \theta_1) = \sum_{i=m+1}^{n} \Big[\sum_{j=1}^{i-1} (n-j)^{-1/2} \Big] (\theta_1 - \theta_0) \quad m=1,2,\ldots n-1 \; \theta_1 > \theta_0).$$

and variance $\mathrm{Var}\, [T^{(1)} | H_0]$.

Since $Z^{(1)} = [T^{(1)} - \mu^{(1)}(m, \theta_1)] / \sigma(T^{(1)})$ is distributed as a standard normal, H_0 is rejected whenever

$$Z^{(1)} \geq Z_\alpha \tag{2.2}$$

and the power of the test when m and θ_1 are the shift point and new mean is

$$\beta_m^{(1)}(\theta_1) = P[Z \geq z_\alpha - \mu^{(1)}(m,\theta_1)/\sigma(T^{(1)})], \tag{2.3}$$

where z_α is the upper 100α percent point of n (0,1) and $\sigma(T^{(1)})$ is the standard deviation of $T^{(1)}$.

If the initial mean is unknown, a size α test of H_0 versus H_1 is to reject H_0 whenever

$$Z^{(2)} \geq z_\alpha$$

where

$$Z^{(2)} = [T^{(2)} - \mu^{(2)}(m,\theta_0,\theta_1)]/\sigma(T^{(2)})$$

with

$$T^{(2)} = \sum_{s-1}^{n-1} (\bar{X}_{s-|-1,n} - \bar{X}_{1,s}) \ [s(n-s)/n]^{1/2}$$

$$= \sum_{i=1}^{n} [\sum_{j=1}^{i-1} (j/n(n-j)) - \sum_{j=i}^{n-1} (n-j)/nj)] \ x_i , \tag{2.4.}$$

and $\mu^{(2)}(m,\theta_0, \theta_1)$ and $\sigma(T^{(2)})$ are the mean and standard deviation of $T^{(2)}$. The power of the test for the alternative (m,θ_0, θ_1) is

$$\beta_m^{(2)}(\theta_0, \theta_1) = P[Z^{(2)} \geq z_\alpha - \mu^{(2)}(m, \theta_0, \theta_1)/\sigma(T^{(2)}) . \tag{2.5.}$$

Tables I and II are tabulations of the power of the tests based on $T^{(1)}$ and $T^{(2)}$ respectively for a size $\alpha = .05$, n=12, m=1,3,...,11, $\theta_0=0$, $\sigma^2=1$, and $\theta_1=.3, .6, .9, \ldots, 1.2$. The power of the Bayes' procedure, chernoff and Zacks (1964), is also tabulated and we see that when θ_0 is know $T^{(1)}$ has higher power if the change occurs early $(1 \leq m \leq 3)$.

The tests based on $T^{(1)}$ and $T^{(2)}$ are unbiased, because $\mu^{(1)}(m,\theta_1)$ and $\mu^{(2)}(m,\theta_0, \theta_1)$ are non-negative for $\theta_1 > \theta_0$ and shifts m=1,2,...,n-1.

3. Two-sided alternatives, $\theta_1 \neq \theta_0$.

When we consider the two-sided alternative, the non-null distribution of the test statistics ($T^{(3)}$ and $T^{(4)}$, corresponding to θ_0 known and unknown, is approximated by equating their first two moments to a weighted chi-square $a\chi^2(b)$ and solving for a and b. Thus the approximate power of the tests are found by interpolating the central chi-square tables or by integrating the gamma density.

If θ_0 is known the likelihood-ratio statistic of H_{0s} versus H_{s1} (s=1,2,...,n-1) is $r^{(s)} = (n\text{-}s)\ (X_{s+1,n} - \theta_0)^2$ and averaging over the shift points gives

$$T^{(3)} = \Sigma\ (n\text{-}s)\ (X_{s+1,n} - \theta_0)^2, \tag{3.1}$$

a sum of correlated chi-square statistics each with one d.f. Letting

$$E[T^{(3)}|H_0] = a_0 b_0$$
$$Var[T^{(3)}|H_0] = 2a^2{}_0 b$$

and solving for a_0 and b_0, we have $a_0 = n/2$ and $b_0 = 2(n\text{-}1)/n$, and H_0 is rejected whenever

$$T^{(3)} \geq a_0\ \chi_\alpha^2\ (b_0) \tag{3.2}$$

where $\chi_\alpha^2\ (b_0)$ is the upper 100α percent point of the χ^2 distribution with b_0 d.f. The approximate power of the test, for all m=1,2,...,n-1 and θ_1, is

$$\beta_m{}^{(3)}(\theta_1) = P[T^{(3)} \geq a_0\ \chi_\alpha^2(b_0)\,|\,(m,\theta_1)].$$

The distribution of $T^{(3)}$ for given alternative (m, θ_1) is found the same way as above by solving

$$E[T^{(3)}|(m,\theta_1)] = a(m,\theta_1)\ b(m,\theta_1) \tag{3.3}$$

and

$$Var[T^{(3)}|(m,\theta_1)] = 2a^2(m.\ \theta_1)\ b(m,\ \theta_1) \tag{3.4}$$

for $a(m,\theta_1)$ and $b(m,\theta_1)$, and using

$$\beta_m{}^{(3)}(\theta_1) = P[a(m,\theta_1)\ \chi^2[b(m,\theta_1)] \geq a_0\chi_\alpha{}^2(b_0)] \tag{3.5}$$

for the power of the size α test based on $A^{(3)}$ (3.2). (3.3) and (3.4) are easily solved by using the fact that

$$E[T^{(3)}|(m,\theta_1)] = \sum_{s=1}^{n-1} E(X'A_sX)$$

$$= \sum_{s=1}^{n-1} T_r(A_s) + \sum_{s=1}^{n-1} \mu' A_s u \qquad (3.6)$$

where T_r is the trace of a matrix and

$$Var[T^{(3)}|(m,\theta_1)] = \sum_{s=1}^{n-1} Var(X,A_sX) + 2\sum_{s<t}\sum cov \quad (X'A_sX, X'A_tX) \qquad (3.7)$$

where $(n\text{-}s)\ X_{s-|-1,n} - \theta_0)^2 = X'A_sX$ and

$$X' = [X_1\text{-}\theta_0), (X_2\text{-}\theta_0), \ldots, (X_n\text{-}\theta_0)]$$

and

$$A_s = (n\text{-}s)^{-1} \left[\begin{array}{c|c} \phi & \phi \\ \hline \phi & J^{n-s}_{ns} \end{array}\right]$$

where the ϕ are a matrices of zeroes and J^{n-s}_{n-s} a (n-s) x (n-s) matrix of ones.

When θ_0 is unknown, the test is based on the statistic

$$T^{(4)} = \sum (X_{s-|-,n} - X_{1,s})^2[s(n\text{-}s)/n], \qquad (3.8)$$

and the critical region and power function found by the method of moments as before.

Tables III and IV gives the approximate power of a size α=.05 test based on $T^{(3)}$ and $T^{(4)}$ for the same set of parameters used for the one-sided alternatives.

TABLE I

THE POWERS OF THE ONE-TAILED ($\theta_1 > \theta_0$) MLRT AND THE BAYES BEST WHEN THE INITIAL MEAN ($\theta_0=0$) AND VARIANCE ($\sigma^2 = 1$) ARE KNOWN FOR n = 12 AND α = .05

θ_1	m	Modified LRT*	Bayes**
.3	1	.2087	.2222
	3	.2002	.2105
	5	.1804	1846
	7	.1502	.1480
	9	.1120	.1066
	11	.0704	.0670
.6	1	.5091	.5459
	3	.4854	.5141
	5	.4276	.4399
	7	.3348	.3283
	9	.2156	.1991
	11	.0967	.0882
.9	1	.8420	.8403
	3	.7786	.8094
	5	.7084	.7243
	7	.5725	.5618
	9	.3600	.3283
	11	.1295	.1141
1.2	1	.9546	.9697
	3	.9420	.9569
	5	8997	.9103
	7	.7858	.7750
	9	.5281	.4822
	11	.1694	.1450

* Obtained from Equation (2.3)

** Chernoff and Zacks (1964)

TABLE II

THE POWERS OF THE ONE-TAILED ($\theta_1 > \theta_0$) MLRT AND THE BAYES TEST WHEN THE INITIAL MEAN IS UNKNOWN AND THE VARIANCE ($\sigma^2=1$) IS KNOWN FOR $n = 12$ AND $\alpha = .05$)

θ_1-θ_0	m	Modified LRT*	Bayes**
.3	1	.0730	.0659
	3	.1011	.0957
	5	.1106	.1139
	7	.1049	.1139
	9	.0878	.0957
	11	.0636	.0659
.6	1	.1035	.0855
	3	.1826	.1666
	5	.2114	.2216
	7	.1940	.2216
	9	.1436	.1666
	11	.0798	.0855
.9	1	.1422	.1092
	3	.2961	.2647
	5	.3520	.3715
	7	.3182	.3715
	9	.2195	.2647
	11	.0991	.1092
1.2	1	.1898	.1372
	3	.4341	.3858
	5	.5167	.5442
	7	.4674	.5442
	9	.3143	.3858
	11	.1216	.1372

* Obtained from Equation (2.5)

** Chernoff and Zacks (1964)

TABLE III

THE POWERS OF THE TWO-TAILED MODIFIED LRT WHEN THE INITIAL MEAN ($\theta_0 = 0$) AND VARIANCE ($\sigma^2 = 1$) ARE KNOWN; AND WHEN THE INITIAL MEAN IS UNKNOWN BUT VARIANCE ($\sigma^2 = 1$) IS KNOWN FOR $n = 12$ AND $\alpha = .05$

θ_1-θ_0	m	MLRT When Initial Mean Is Known*	MLRT When Initial Mean Is Unknown
.3	1	.1453	.0572
	3	.1420	.0765
	5	.1274	.0866
	7	.1045	.0838
	9	.0769	.0698
	11	.0545	.0534
.6	1	.3897	.0764
	3	.3556	.1354
	5	.3029	.1651
	7	.2317	.1617
	9	.1458	.1226
	11	.0658	.0666
.9	1	.7135	.1033
	3	.6471	.2074
	5	.5545	.2652
	7	.4219	.2661
	9	.2518	.2001
	11	.0848	.0865
1.2	1	.9299	.1338
	3	.8810	.2929
	5	.7980	.3886
	7	.6445	.3969
	9	.3940	.3004
	11	.1119	.1137

* Obtained from Equation (3.5)

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